Problem 1 Relativistic Quantum Mechanics

Discussion date during Tutorial: 09 - 16 /02/2023

Consider the one-dimensional harmonic oscillator, with kinetic energy T and potential energy V:

$$T = \tfrac{1}{2} m \dot{x}^2 , \qquad V = \tfrac{1}{2} m \omega^2 x^2 .$$

The Lagrangian is $L = T - V$.

a) What is the Euler-Lagrange equation for the harmonic oscillator?

b) What is the canonical momentum p?

c) What is the Hamiltonian $H(p, x)$?

d) Determine the Hamilton equations of motion and show that they are equivalent to the equation derived in (a).

e) Use the result of (b) to write L as a function of p and x. Calculate the Poisson bracket $G = \{L,\, H\}_{\text{PB}}$. Calculate also $\{G,\, H\}_{\text{PB}}$.

f) Let

$$a = \frac{1}{\sqrt{2m\omega}}\,(p + im\omega x), \qquad a^* = \frac{1}{\sqrt{2m\omega}}\,(p - im\omega x) .$$

Express H in terms of a and a^*. Calculate $\{a,\, a^*\}_{\text{PB}}$. Express $\{a,\, H\}_{\text{PB}}$ and $\{a^*,\, H\}_{\text{PB}}$ in terms of a and/or a^*.

Now consider a field $\phi(x)$ with Lagrangian

$$L = \tfrac{1}{2}\eta^{\mu\nu}(\partial_\mu \phi)(\partial_\nu \phi) - \tfrac{1}{2} m^2 \phi^2$$

g) Derive the corresponding expressions of exercises (a) up to and including (d) for this field $\phi(x)$.

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Problem 2 Relativistic Quantum Mechanics

x^μ stands for the four coordinates ct, x^1, x^2, x^3. The derivatives ∂_μ stand for the partial derivatives $\frac{\partial}{\partial x^\mu}$. If an expression contains two equal indices one should be covariant (lower), the other contravariant (upper). Summation over such indices is implicitly assumed. Indices can be raised (lowered) using the Minkowski metric $\eta^{\mu\nu}$ ($\eta_{\mu\nu}$), which is a diagonal matrix with entries $+1, -1, -1, -1$.

(a) We set $(x)^2 = x^\mu x_\mu$, $(x)^4 = (x)^2(x)^2$. Evaluate:

$$\partial_\mu (x)^2\,, \quad \partial^\lambda \partial_\lambda (x^\mu x^\rho)\,, \quad \partial_\mu \partial_\nu (x)^4\,.$$

(b) The motion of a complex field $\psi(x)$ is governed by the Lagrangian

$$\mathcal{L} = \partial_\mu \psi^* \partial^\mu \psi - m^2 \psi^* \psi - \frac{\lambda}{2}(\psi^* \psi)^2.$$

Write down the Euler-Lagrange field equations for this system. Verify that the Lagrangian is invariant under the infinitesimal transformation

$$\delta\psi = i\alpha\psi \qquad , \qquad \delta\psi^* = -i\alpha\psi^*\,.$$

Derive the Noether current associated with this transformation and verify explicitly that it is conserved using the field equations satisfied by ψ.

(c) A Lorentz transformation $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$ is such that it preserves the Minkowski metric $\eta_{\mu\nu}$, meaning that $\eta_{\mu\nu} x^\mu x^\nu = \eta_{\mu\nu} x'^\mu x'^\nu$ for all x. Show that this implies that

$$\eta_{\mu\nu} = \eta_{\sigma\tau} \Lambda^\sigma{}_\mu \Lambda^\tau{}_\nu.$$

Use this result to show that an infinitesimal transformation of the form

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$$

is a Lorentz transformation when $\omega^{\mu\nu}$ is anti-symmetric: i.e. $\omega^{\mu\nu} = -\omega^{\nu\mu}$.

Write down the matrix form for $\omega^\mu{}_\nu$ that corresponds to a rotation through an infinitesimal angle θ around the x^3-axis. Do the same for a boost along the x^1-axis by an infinitesimal velocity v.

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Problem Sheet 3 — Relativistic Quantum Mechanics

In this problem sheet we apply the Noether theorem to spacetime symmetries.

a/ Review the derivation of the energy momentum tensor $T_{\mu\nu}$ as the Noether current of translation symmetry $x^\mu \to x^\mu - \epsilon^\mu$ (see section 1.3.2 in the D. Tong notes).

b/ Derive the explicit expression for $T_{\mu\nu}$ for the free real scalar (1.7), the Schrödinger Lagrangian (1.15), and the Maxwell Lagrangian (1.18). Are the explicit expressions symmetric $T_{\mu\nu} = T_{\nu\mu}$? What can you say about the trace of the energy momentum tensor $T = \eta^{\mu\nu} T_{\mu\nu}$ in each of the examples?

c/ Let us now consider Lorentz transformations as derived in the previous problem set $x^\mu \to x^\mu + \omega^\mu{}_\nu x^\nu$. For simplicity, we restrict to scalar fields $\phi(x)$ with Lagrangian density (1.7): $\mathcal{L} = 1/2(\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\,\phi^2)$. Show that

$$\phi(x) \to \phi'(x) = \phi(x) - \omega^\mu{}_\nu x^\nu \partial_\mu \phi(x)\,.$$

d/ Use this result to show that the Lagrangian density (1.7) transforms as $\mathcal{L} \to \mathcal{L} + \delta\mathcal{L}$, with

$$\delta\mathcal{L} = -\partial_\mu \left(\omega^\mu{}_\nu x^\nu \mathcal{L}\right)$$

e/ Using Noether's theorem deduce the existence of the conserved current

$$j^\mu = -\omega^\rho{}_\nu \left[T^\mu{}_\rho x^\nu\right]$$

and show that indeed $\partial_\mu j^\mu = 0$. Can you give an interpretation to the conserved charges $Q = \int_{\mathbf{R}^3} d^3x\, j^0$?

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Problem Sheet 4 — Relativistic Quantum Mechanics

In this problem set we consider the canonical quantization of a real scalar field ϕ with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \,,$$

and study aspects of the quantum theory.

a/ The Fourier decomposition of a real scalar field and its conjugate momentum in the Schrödinger picture is given by

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[a_{\vec{p}}\, \mathrm{e}^{\mathrm{i}\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}}\, \mathrm{e}^{-\mathrm{i}\vec{p}\cdot\vec{x}} \right]$$
$$\pi(\vec{x}) = -\mathrm{i} \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{E_{\vec{p}}}{2}} \left[a_{\vec{p}}\, \mathrm{e}^{\mathrm{i}\vec{p}\cdot\vec{x}} - a^\dagger_{\vec{p}}\, \mathrm{e}^{-\mathrm{i}\vec{p}\cdot\vec{x}} \right]$$

Show that the commutation relations

$$[\phi(\vec{x}), \phi(\vec{y})] = [\pi(\vec{x}), \pi(\vec{y})] = 0 \quad \text{and} \quad [\phi(\vec{x}), \pi(\vec{y})] = \mathrm{i}\delta^{(3)}(\vec{x} - \vec{y})$$

imply that

$$[a_{\vec{p}}, a_{\vec{q}}] = \left[a^\dagger_{\vec{p}}, a^\dagger_{\vec{q}}\right] = 0 \quad \text{and} \quad \left[a_{\vec{p}}, a^\dagger_{\vec{q}}\right] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q})$$

b/ You have seen in the lecture that the normal ordered Hamiltonian takes the form $H = (2\pi)^{-3} \int d^3p\, E_{\vec{p}}\, a^\dagger_{\vec{p}} a_{\vec{p}}$ (section 2.3 in the notes). Show that

$$\left[H, a^\dagger_{\vec{p}}\right] = E_{\vec{p}}\, a^\dagger_{\vec{p}} \,, \qquad \text{and} \qquad \left[H, a_{\vec{p}}\right] = -E_{\vec{p}}\, a_{\vec{p}} \,,$$

and use it to derive the energy of an n–particle state

$$|\vec{p}_1, \vec{p}_2, \cdots, \vec{p}_n\rangle = a^\dagger_{\vec{p}_1} \cdots a^\dagger_{\vec{p}_n} |0\rangle \,.$$

c/ Similarly, using the operator expansion given above, one can derive the total momentum operator $\vec{P} = (2\pi)^{-3} \int d^3p\, \vec{p}\, a^\dagger_{\vec{p}} a_{\vec{p}}$ (see eq. (2.45) in the notes). Find the total momentum of a an n–particle state.

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Problem Sheet 5 — Relativistic Quantum Mechanics

a/ We wish to quantize the theory defined by the following Lagrangian:

$$\mathcal{L} = +i\psi^\star \partial_0 \psi - \frac{1}{2m} \vec{\nabla}\psi^\star \vec{\nabla}\psi$$

We will work in the Schrödinger picture. Explain why the correct commutation relations are

$$[\psi(\vec{x}), \psi(\vec{y})] = 0 = \left[\psi^\dagger(\vec{x}), \psi^\dagger(\vec{y})\right] , \quad \text{and} \quad \left[\psi(\vec{x}), \psi^\dagger(\vec{y})\right] = \delta^{(3)}(\vec{x}-\vec{y})$$

Expand the fields in a Fourier decomposition as

$$\psi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} a_{\vec{p}}\, e^{\vec{p}\cdot\vec{x}}$$

$$\psi^\dagger(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} a^\dagger_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}}$$

Determine the commutation relations obeyed by $a_{\vec{p}}$ and $a^\dagger_{\vec{p}}$. Why do we have only a single set of creation and annihilation operators $a_{\vec{p}}$, $a^\dagger_{\vec{p}}$ even though ψ is complex? What is the physical significance of this fact? Show that one particle states have the energy appropriate to a free non-relativistic particle of mass m.

b/ Determine the numbers a_1, a_2, a_3 :

$$\gamma^\mu \gamma^\rho \gamma_\mu = a_1 \gamma^\rho, \quad \eta_{\mu\nu}\gamma^\rho\gamma^\mu\gamma_\rho\gamma^\nu = a_2, \quad [\gamma_\mu, \gamma^\nu]\,\gamma^\mu = a_3 \gamma^\nu ,$$

where all you have to use is $\{\gamma_\mu, \gamma_\nu\} = \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\,\eta_{\mu\nu}\mathbb{1}_4$, where $\mathbb{1}_4$ denotes the 4×4 unit matrix.

c/ The Weyl representation of the Clifford algebra is given by,

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} \quad , \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

where σ^i with $i = 1, 2, 3$ denote the Pauli matrices, and $\mathbb{1}_2$ denotes the 2×2 unit matrix. Show that these indeed satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\,\eta^{\mu\nu}\mathbb{1}_4$. Find a unitary matrix U such that $(\gamma')^\mu = U\gamma^\mu U^\dagger$, where $(\gamma')^\mu$ form the Dirac representation of the Clifford algebra

$$(\gamma')^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \quad , \quad (\gamma')^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} .$$

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Problem Sheet 6 — Relativistic Quantum Mechanics

c/ Consider the Dirac action

$$S = \int d^4x \, \bar{\psi} \, (\mathrm{i}\gamma^\mu \partial_\mu - m) \, \psi \, ,$$

and derive the equations of motion for ψ and $\bar{\psi}$. Show that both equations are related by Dirac conjugation $\bar{\psi} = \psi^\dagger \gamma^0$.

d/ Show that $V^\mu = \bar{\psi}\gamma^\mu\psi$ is a Noether current, and hence conserved. What is the corresponding symmetry?

e/ The matrix γ^5 is defined as

$$\gamma^5 = -\mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 \tag{4}$$

and satisfies

$$\{\gamma^5, \gamma^\mu\} = 0. \tag{5}$$

Show that the combinations $A = \bar{\psi}\gamma^5\psi$ and $A^\mu = \bar{\psi}\gamma^5\gamma^\mu\psi$ also transform under infinitesimal Lorentz transformations as a scalar and vector, respectively:

$$\delta A = 0 \, , \qquad \delta A^\mu = \Omega^\mu{}_\nu A^\nu \, . \tag{6}$$

f/ Show that the axial current $A^\mu = \bar{\psi}\gamma^5\gamma^\mu\psi$ is conserved if $m = 0$. What is the corresponding symmetry?

g/ Using the fact that there are 16 independent combinations $\bar{\psi}^\alpha\psi_\beta$ $(\alpha\,,\beta = 1,2,3,4)$ quadratic in the spinor field $\psi(x)$ and that so far we found only 10 such independent combinations, namely $(S, V^\mu, A$ and $A^\mu)$, we observe that there are 6 independent combinations missing. Consider the following 16 additional combinations:

$$\bar{\psi}\gamma^\mu\gamma^\nu\psi\,. \tag{7}$$

Argue which ones of these 16 combinations are equivalent to the ones we already have and which ones could provide the 6 missing combinations.